The derived category of (commutative) DG algebras

Teresa Conde

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Notation

- $\mathbb{K}$ – commutative ring with identity
- $\mathcal{B}$ – category; $\mathrm{Ob}\,\mathcal{B}$ – objects; $\mathrm{Mor}_{\mathcal{B}}\,(M, N)$ – morphisms
- "DG" = "differential graded"

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Outline

1. The category of complexes $\mathcal{DGM}(\mathbb{K})$
2. $(\mathcal{DGM}(\mathbb{K}), \otimes)$ as a closed symmetric monoidal category
3. DG algebras and DG modules
4. The category $\mathcal{DGM}(A)$ and the functors $- \otimes_A -$, $\mathrm{Hom}_A(-,-)$
5. The homotopy category $\mathcal{H}(A)$
6. The derived category $\mathcal{D}(A)$
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1. The category of complexes $\mathcal{DGM}(\mathbb{K})$

1.1. Objects and morphisms in $\mathcal{DGM}(\mathbb{K})$

The objects of $\mathcal{DGM}(\mathbb{K})$ are *complexes of $\mathbb{K}$-modules*, i.e. sequences of homomorphisms of $\mathbb{K}$-modules

$$M = \quad \cdots \longrightarrow M_{i+1} \xrightarrow{\partial_{i+1}^M} M_i \xrightarrow{\partial_i^M} M_{i-1} \longrightarrow \cdots$$

such that $\partial_i^M \circ \partial_{i+1}^M = 0$ for all i.

We write $m \in M$ if $m \in M_d$ for a certain d. In this case we say that m has *degree* d, and write $|m| = d$.

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We write $m \in M$ if $m \in M_d$ for a certain d. In this case we say that m has *degree* d, and write $|m| = d$.

A morphism $\beta : M \longrightarrow N$ in $\mathcal{DGM}(\mathbb{K})$ is a family of homomorphisms of $\mathbb{K}$-modules

$$\beta = (\beta_i : M_i \longrightarrow N_i)_{i \in \mathbb{Z}}$$

such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M_{i+1} \arrow[r, "\partial^M_{i+1}"] \arrow[d, "\beta_{i+1}"] & M_i \arrow[r, "\partial^M_i"] \arrow[d, "\beta_i"] & M_{i-1} \arrow[r] \arrow[d, "\beta_{i-1}"] & \cdots \\
\cdots \arrow[r] & N_{i+1} \arrow[r, "\partial^N_{i+1}"] & N_i \arrow[r, "\partial^N_i"] & N_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

The category $\mathcal{DGM}(\mathbb{K})$ is a $\mathbb{K}$-category which is complete and cocomplete.

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1.2. Some functors and further notions

Given a complex M

$$M = \quad \cdots \longrightarrow M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \longrightarrow \cdots \quad ,$$

we may consider a new complex $M^\natural$

$$M^\natural = \quad \cdots \longrightarrow M_{i+1} \xrightarrow{0} M_i \xrightarrow{0} M_{i-1} \longrightarrow \cdots \quad .$$

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Define $\mathcal{GM}(\mathbb{K})$ to be the full subcategory of $\mathcal{DGM}(\mathbb{K})$ whose objects are the complexes M such that $\partial^M = 0$.

Then $(-)^\natural$ defines a forgetful functor

$$\begin{aligned}(-)^\natural : \mathcal{DGM}(\mathbb{K}) &\longrightarrow \mathcal{GM}(\mathbb{K})\\ M &\longmapsto M^\natural.\end{aligned}$$

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We may also consider the autofunctor Σ

$$\Sigma : \mathcal{DGM}\left(\mathbb{K}\right) \longrightarrow \mathcal{DGM}\left(\mathbb{K}\right)$$

which assigns a complex $\Sigma\left(M\right)$ to each complex M, where

$$\begin{aligned} \left(\Sigma\left(M\right)\right)_i &= M_{i-1}, \\ \partial_i^{\Sigma(M)} &= -\partial_{i-1}^{M}. \end{aligned}$$

The functors Σ and $(-)^\natural$ commute with each other.

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The functors Σ and $(-)^\natural$ commute with each other.

Let M and N be complexes.

We call β a *chain map* of degree d from M to N if $\beta \in \mathrm{Mor}_{\mathcal{DGM}(\mathbb{K})}\left(M, \Sigma^{-d}(N)\right)$.

We call β a *homomorphism* of degree d from M to N if β is a chain map of degree d from $M^\natural$ to $N^\natural$. Write $|\beta| = d$ and define

$$\mathrm{Hom}(M, N) = \coprod_{i:\, i \in \mathbb{Z}} \mathrm{Mor}_{\mathcal{GM}(\mathbb{K})}\left(M^\natural, \Sigma^i N^\natural\right).$$

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1.3. A tensor product in $\mathcal{DGM}(\mathbb{K})$

A *graded set* is a family of sets $(X_i)_{i\in\mathbb{Z}}$.

A *homogeneous map* of graded sets, $\beta : X \longrightarrow Y$, is a family of maps

$$\beta = (\beta_i : X_i \longrightarrow Y_{i+d})_{i\in\mathbb{Z}},$$

for some fixed d.

If X and Y are graded sets their *graded product* is the graded set

$$X \boxtimes Y = \left(\bigsqcup_{i,j:\, i+j=h} (X_i \times Y_j) \right)_{h\in\mathbb{Z}}.$$

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Let L, M and N be in $\mathrm{Ob}\,\mathcal{GM}\,(\mathbb{K})$.

A homogeneous map $\psi : L \boxtimes M \longrightarrow N$ is called $\mathbb{K}$-*bilinear* if, for every $i, j \in \mathbb{Z}$ with $i + j = h$, $l, l' \in L_i$, $m, m' \in M_j$ and $k \in \mathbb{K}$, there are identities

$$\begin{aligned} \psi_h\left(l + l', m\right) &= \psi_h\left(l, m\right) + \psi_h\left(l', m\right), \\ \psi_h\left(l, m + m'\right) &= \psi_h\left(l, m\right) + \psi_h\left(l, m'\right), \\ \psi_h\left(kl, m\right) &= \psi_h\left(l, km\right). \end{aligned}$$

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Let L, M and N be in $\mathrm{Ob}\,\mathcal{GM}\,(\mathbb{K})$.

Denote by $L \otimes M$ the graded module over $\mathbb{K}$ with h^{th} component

$$(L \otimes M)_h = \bigoplus_{i,j:\, i+j=h} (L_i \otimes_{\mathbb{K}} M_j)\,.$$

Universal property of $\otimes$

Let $\psi : L \boxtimes M \longrightarrow N$ be a homogeneous $\mathbb{K}$-bilinear map of graded sets of degree d. There exists a unique homomorphism of complexes

$$\psi' : L \otimes M \longrightarrow N$$

such that:

- $|\psi'| = d$,
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Let M, M', L and L' be in $\operatorname{Ob}\mathcal{GM}(\mathbb{K})$ and consider the homomorphisms of complexes $\lambda : L \longrightarrow L'$ and $\mu : M \longrightarrow M'$.

By the universal property of $\otimes$ there is a homomorphism

$$\lambda \otimes \mu : L \otimes M \longrightarrow L' \otimes M'$$

of degree $|\lambda| + |\mu|$, satisfying

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In particular, consider the homomorphism of degree -1 given by

$$\partial^{L\otimes M} = \partial^L \otimes \mathrm{id}_M + \mathrm{id}_L \otimes \partial^M : L \otimes M \longrightarrow L \otimes M.$$

We have $\left(\partial^{L\otimes M}\right)^2 = 0$.

Definition

Let L and N be complexes. The *tensor product* $L \otimes M$ is the complex given by the graded module $L^\natural \otimes M^\natural$, endowed with the differential $\partial^{L\otimes M}$.

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Definition

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The tensor product $\otimes$ defines a functor

$$- \otimes - : \mathcal{DGM}(\mathbb{K}) \times \mathcal{DGM}(\mathbb{K}) \longrightarrow \mathcal{DGM}(\mathbb{K})$$

with nice properties.

2. $\mathcal{DGM}(\mathbb{K})$ as a closed symmetric monoidal category

2.1. Monoidal categories

A *monoidal category* $\mathcal{B} = (\mathcal{B}, -\Box-, E, \alpha, \lambda, \rho)$ is a category $\mathcal{B}$ endowed with a functor

$$-\Box- : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B} \text{ (the tensor product)},$$

an object $E \in \mathrm{Ob}\,\mathcal{B}$ (the tensor unit), and three natural isomorphisms,

$$\alpha : (-\Box-) \circ ((-\Box-) \times \mathrm{id}_{\mathcal{B}}) \longrightarrow (-\Box-) \circ (\mathrm{id}_{\mathcal{B}} \times (-\Box-)) \quad \text{(the associator)},$$

$$\lambda : (-\Box-) \circ (E \times \mathrm{id}_{\mathcal{B}}) \longrightarrow \mathrm{id}_{\mathcal{B}} \quad \text{(the left unitor)},$$

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satisfying properties 1 and 2.

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2.1. Monoidal categories

A *monoidal category* $\mathcal{B} = (\mathcal{B}, -\Box-, E, \alpha, \lambda, \rho)$ is a category $\mathcal{B}$ endowed with a functor

$$-\Box- : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B} \text{ (the tensor product)},$$

an object $E \in \mathrm{Ob}\,\mathcal{B}$ (the tensor unit), and three natural isomorphisms,

$$\alpha : (-\Box-) \circ ((-\Box-) \times \mathrm{id}_{\mathcal{B}}) \longrightarrow (-\Box-) \circ (\mathrm{id}_{\mathcal{B}} \times (-\Box-))$$
$$\text{(the associator)},$$

$$\lambda : (-\Box-) \circ (E \times \mathrm{id}_{\mathcal{B}}) \longrightarrow \mathrm{id}_{\mathcal{B}} \text{ (the left unitor)},$$

$$\rho : (-\Box-) \circ (\mathrm{id}_{\mathcal{B}} \times E) \longrightarrow \mathrm{id}_{\mathcal{B}} \text{ (the right unitor)},$$

satisfying properties 1 and 2.

1. The pentagonal diagram commutes for all $A, B, C, D \in \mathrm{Ob}\,\mathcal{B}$

$$\begin{tikzcd}
 & & (A\Box B)\Box(C\Box D) \arrow[drr, "\alpha_{(A\Box B,C,D)}"] & & \\
A\Box(B\Box(C\Box D)) \arrow[urr, "\alpha_{(A,B,C\Box D)}"] \arrow[dr, "\mathrm{id}_A\Box\alpha_{(B,C,D)}"] & & & & ((A\Box B)\Box C)\Box D \\
 & A\Box((B\Box C)\Box D) \arrow[rr, "\alpha_{(A,B\Box C,D)}"] & & (A\Box(B\Box C))\Box D \arrow[ur, "\alpha_{(A,B,C)}\Box\mathrm{id}_D"] &
\end{tikzcd} .$$

2. For $A, B \in \operatorname{Ob}\mathcal{B}$ the triangle identity holds

$$
\begin{tikzcd}
A\Box(E\Box B) \arrow[rr, "\alpha_{A,E,B}"] \arrow[dr, "\mathrm{id}_A\Box\lambda_B"'] & & (A\Box E)\Box B \arrow[dl, "\rho_A\Box\mathrm{id}_B"] \\
& A\Box B &
\end{tikzcd}
.
$$

The category $\mathcal{DGM}(\mathbb{K})$ is a monoidal category with

- tensor product $-\otimes-$,
- tensor unit $\mathbb{K}$,
- α, λ and ρ as expected.

2.2. Symmetric monoidal categories

A monoidal category $\mathcal{B} = (\mathcal{B}, -\Box-, E, \alpha, \lambda, \rho)$ is *symmetric* if it is endowed with a natural isomorphism γ, called the braiding

$$\gamma : (-\Box-) \longrightarrow (-\Box-) \circ (- \times {}^{op} -),$$

satisfying conditions 1, 2 and 3.

1. $\gamma_{(A,B)} \circ \gamma_{(B,A)} = \mathrm{id}_{A\Box B}$, for every $A, B \in \mathrm{Ob}\,\mathcal{B}$.

2. For every $A \in \mathrm{Ob}\,\mathcal{B}$, there is a commutative diagram

$$\begin{tikzcd} A\Box E \arrow[rr, "\gamma_{(A,E)}"] \arrow[dr, "\rho_A"'] & & E\Box A \arrow[dl, "\lambda_A"] \\ & A & \end{tikzcd}.$$

3. For every $A, B, C \in \mathrm{Ob}\,\mathcal{B}$, there is a commutative diagram

$$\begin{array}{ccccc} A\Box(B\Box C) & \xrightarrow{\alpha_{(A,B,C)}} & (A\Box B)\Box C & \xrightarrow{\gamma_{(A\Box B,C)}} & C\Box(A\Box B) \\ {\scriptstyle \mathrm{id}_A\Box\gamma_{(B,C)}}\downarrow & & & & \downarrow{\scriptstyle \alpha_{(C,A,B)}} \\ A\Box(C\Box B) & \xrightarrow{\alpha_{(A,B,C)}} & (A\Box C)\Box B & \xrightarrow{\gamma_{(A,C)}\Box\,\mathrm{id}_B} & (C\Box A)\Box B \end{array}.$$

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A \Box (C \Box B) \arrow[r, "\alpha_{(A,B,C)}"] & (A \Box C) \Box B \arrow[r, "\gamma_{(A,C)} \Box \mathrm{id}_B"] & (C \Box A) \Box B
\end{tikzcd} .$$

The monoidal category $\mathcal{DGM}(\mathbb{K}) = (\mathcal{DGM}(\mathbb{K}), - \otimes -, \mathbb{K}, \alpha, \lambda, \rho)$ is symmetric, with braiding

$$\begin{aligned} \gamma_{(L,M)} : L \otimes M &\longrightarrow M \otimes L \\ l \otimes m &\longmapsto (-1)^{|l||m|} m \otimes l, \end{aligned}$$

for $L, M \in \mathrm{Ob}\, \mathcal{DGA}(\mathbb{K})$.

2.3. Closed symmetric monoidal categories

A symmetric monoidal category $\mathcal{B}$ is *closed* if for all $A \in \operatorname{Ob}\mathcal{B}$, each functor

$$-\Box A : \mathcal{B} \longrightarrow \mathcal{B}$$

has a right adjoint

$$[A, -] : \mathcal{B} \longrightarrow \mathcal{B}.$$

The functor

$$- \otimes M : \mathcal{DGM}\left(\mathbb{K}\right) \times \mathcal{DGM}\left(\mathbb{K}\right) \longrightarrow \mathcal{DGM}\left(\mathbb{K}\right)$$

has a right adjoint, i.e. $\mathcal{DGM}\left(\mathbb{K}\right)$ is a closed symmetric monoidal category.

Let $\mu : M' \longrightarrow M$ and $\nu : N \longrightarrow N'$ be homomorphisms of complexes.

Recall: $\text{Hom}\,(M, N)$ and $\text{Hom}\,(M', N')$ are graded modules over $\mathbb{K}$.

And

$$\text{Hom}\,(\mu, \nu) : \text{Hom}\,(M, N) \longrightarrow \text{Hom}\,\left(M', N'\right)$$

$$\left(\text{Hom}\,(\mu, \nu)\right)(\beta) = (-1)^{|\mu|(|\nu|+|\beta|)}\,\nu \circ \beta \circ \mu$$

is a homomorphism of graded modules over $\mathbb{K}$.

The graded module $\text{Hom}\,(M, N)$, together with

$$\partial^{\text{Hom}\,(M,N)} = \text{Hom}\,\left(\text{id}_M, \partial^N\right) - \text{Hom}\,\left(\partial^M, \text{id}_N\right),$$

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becomes a complex.

Indeed,

$$\mathrm{Hom}\left(-,-\right): \mathcal{DGM}\left(\mathbb{K}\right)^{op} \times \mathcal{DGM}\left(\mathbb{K}\right) \longrightarrow \mathcal{DGM}\left(\mathbb{K}\right),$$

defines a functor, and there is a natural isomorphism

$$\mathrm{Mor}_{\mathcal{DGM}(\mathbb{K})}\left(L \otimes M, N\right) \cong \mathrm{Mor}_{\mathcal{DGM}(\mathbb{K})}\left(L, \mathrm{Hom}\left(M, N\right)\right).$$

Indeed,

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3. DG algebras and DG modules

3.1. Monoids / DG algebras

A *monoid* in a monoidal category $\mathcal{B} = (\mathcal{B}, \Box, E, \alpha, \lambda, \rho)$ is an object $A \in \operatorname{Ob} \mathcal{B}$, together with two morphisms

$$\mu : A \Box A \longrightarrow A \text{ (multiplication)},$$

$$\eta : E \longrightarrow A \text{ (unit)},$$

such that diagrams 1 and 2 commute.

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such that diagrams 1 and 2 commute.

1.

$$
\begin{tikzcd}
A\Box(A\Box A) \arrow[r, "\alpha_{(A,A,A)}"] \arrow[d, "\mathrm{id}_A \Box\mu"'] & (A\Box A)\Box A \arrow[r, "\mu\Box \mathrm{id}_A"] & A\Box A \arrow[d, "\mu"] \\
A\Box A \arrow[rr, "\mu"] & & A
\end{tikzcd}
$$

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$$
\begin{tikzcd}
E\Box A \arrow[r, "\eta\Box \mathrm{id}_A"] \arrow[dr, "\lambda_A"'] & A\Box A \arrow[d, "\mu"] & A\Box E \arrow[l, "\mathrm{id}_A \Box\eta"'] \arrow[dl, "\rho_A"] \\
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\end{tikzcd}$$

If $\mathcal{B}$ is a symmetric monoidal category (with braiding γ), then a monoid $A = (A, \mu, \eta)$ in $\mathcal{B}$ is said to be *commutative* if the diagram

$$\begin{tikzcd} A\Box A \arrow[r, "\gamma_{(A,A)}"] \arrow[dr, "\mu"'] & A\Box A \arrow[d, "\mu"] \\ & A \end{tikzcd}$$

commutes.

Given a monoid $A = (A, \mu, \eta)$ we may form the *opposite monoid* $A^{op} = (A, \mu \circ \gamma_{(A,A)}, \eta)$.

A commutative $\Rightarrow A = A^{op}$

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$$\boxed{A \text{ commutative} \Rightarrow A = A^{op}}$$

Definition

1. A complex A in $\mathcal{DGM}(\mathbb{K})$ is called a *DG algebra* if it is a monoid in $(\mathcal{DGM}(\mathbb{K}), - \otimes -, \mathbb{K}, \alpha, \lambda, \rho)$.
2. A DG algebra A is said to be *commutative* if it is commutative monoid in $(\mathcal{DGM}(\mathbb{K}), - \otimes -, \mathbb{K}, \alpha, \lambda, \rho)$ with respect to the braiding γ.

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3.2. Modules / DG modules

Let $\mathcal{B} = (\mathcal{B}, \Box, E, \alpha, \lambda, \rho)$ be a monoidal category. A *(left) module* B over a monoid $A = (A, \mu, \eta)$ is an object B in $\mathcal{B}$, together with a morphism

$$\nu : A \Box B \longrightarrow B$$

such that the diagrams

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A\Box(A\Box B) \arrow[r, "\alpha_{(A,A,B)}"] \arrow[d, "\mathrm{id}_A \Box \nu"'] & (A\Box A)\Box B \arrow[r, "\mu\Box \mathrm{id}_B"] & A\Box B \arrow[d, "\nu"] \\
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Right modules over a monoid are defined symmetrically.

Let $\mathcal{B}$ be a symmetric monoidal category, with braiding γ, and let A be a monoid.

$$B = (B, \nu) \text{ is a left module over } A = (A, \mu, \eta)$$
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$$B' = \left(B, \nu \circ \gamma_{(B,A)}\right) \text{ is a right module over } A^{op}$$

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Consider $\beta : B \longrightarrow B'$, morphism in $\mathcal{B}$, with $B = (B, \nu)$, $B' = (B', \nu')$ modules over A.

We call β a *morphism of modules* over A if the diagram

$$\begin{array}{ccc} A \Box B & \xrightarrow{A \Box \beta} & A \Box B' \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ B & \xrightarrow{\beta} & B' \end{array}$$

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Definition

Let A be a DG algebra.

1. A complex M in $\mathcal{DGM}(\mathbb{K})$ is a *DG module* over A if M is a module over the monoid A in the monoidal category $\mathcal{DGM}(\mathbb{K})$.
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4. The category $\mathcal{DGM}(A)$ and the functors $- \otimes_A -$, $\mathrm{Hom}_A(-,-)$

Let A be a DG algebra.

The DG modules over A and the morphisms of DG modules over A form a subcategory of $\mathcal{DGM}(\mathbb{K})$: denote it by $\mathcal{DGM}(A)$.

The category $\mathcal{DGM}(A)$ is a $\mathbb{K}$-category which is complete and cocomplete.

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4.1 The functor $\mathrm{Hom}_A(-,-)$

Let M and N be DG modules over a DG algebra A. We have morphisms of DG algebras

$$\phi^M : A \longrightarrow \mathrm{Hom}(M, M),$$

$$\phi^N : A \longrightarrow \mathrm{Hom}(N, N).$$

Consider the morphisms of complexes

$$f : \mathrm{Hom}(M, N) \xrightarrow{\mathrm{Hom}(M,-)_{M,N}} \mathrm{Hom}(\mathrm{Hom}(M, M), \mathrm{Hom}(M, N)) \xrightarrow{\mathrm{Hom}(\phi^M, \mathrm{Hom}(M,N))} \mathrm{Hom}(A, \mathrm{Hom}(M, N)),$$

$$g : \mathrm{Hom}(M, N) \xrightarrow{\mathrm{Hom}(-,N)_{M,N}} \mathrm{Hom}(\mathrm{Hom}(N, N), \mathrm{Hom}(M, N)) \xrightarrow{\mathrm{Hom}(\phi^N, \mathrm{Hom}(M,N))} \mathrm{Hom}(A, \mathrm{Hom}(M, N))$$

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$$g : \mathrm{Hom}\,(M, N) \xrightarrow{\mathrm{Hom}\,(-,N)_{M,N}} \mathrm{Hom}\,(\mathrm{Hom}\,(N, N)\,, \mathrm{Hom}\,(M, N)) \xrightarrow{\mathrm{Hom}\,(\phi^N, \mathrm{Hom}\,(M,N))} \mathrm{Hom}\,(A, \mathrm{Hom}\,(M, N))\,.$$

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$$f : \mathrm{Hom}(M,N) \xrightarrow{\mathrm{Hom}(M,-)_{M,N}} \mathrm{Hom}(\mathrm{Hom}(M,M), \mathrm{Hom}(M,N)) \xrightarrow{\mathrm{Hom}(\phi^M, \mathrm{Hom}(M,N))} \mathrm{Hom}(A, \mathrm{Hom}(M,N)),$$

$$g : \mathrm{Hom}(M,N) \xrightarrow{\mathrm{Hom}(-,N)_{M,N}} \mathrm{Hom}(\mathrm{Hom}(N,N), \mathrm{Hom}(M,N)) \xrightarrow{\mathrm{Hom}(\phi^N, \mathrm{Hom}(M,N))} \mathrm{Hom}(A, \mathrm{Hom}(M,N)).$$

4.1 The functor $\mathrm{Hom}_A(-,-)$

Let M and N be DG modules over a DG algebra A. We have morphisms of DG algebras

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Define $\mathrm{Hom}_A(M, N)$ to be the equaliser of the morphisms f and g.

The functor $\mathrm{Hom}(-,-)$ restricts to a functor

$$\mathrm{Hom}_A(-,-) : \mathcal{DGM}(A)^{op} \times \mathcal{DGM}(A) \longrightarrow \mathcal{DGM}(\mathbb{K}).$$

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4.2 The functor $- \otimes_A -$

Let L and M be DG modules over the DG algebras A^{op} and A, respectively, with actions

$$\nu^L : A \otimes L \longrightarrow L,$$

$$\nu^M : A \otimes M \longrightarrow M.$$

Consider the morphisms of complexes

$$f : A \otimes L \otimes M \xrightarrow{\nu^L \otimes M} L \otimes M ,$$

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If L and M are DG modules over A, we may consider the complex

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5. The homotopy category $\mathcal{H}(A)$

Let A be a DG algebra and

$$\beta : M \longrightarrow N$$

a morphisms of *DG* modules over A.

Say that β is *null homotopic* if

$$\beta = \partial^N \circ \chi + \chi \circ \partial^M,$$

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The *homotopy category of a DG algebra* A, denoted by $\mathcal{H}(A)$ is the category defined by:

- $\operatorname{Ob}\mathcal{H}(A) = \operatorname{Ob}\mathcal{DGM}(A)$,
- $\operatorname{Mor}_{\mathcal{H}(A)}(M, N) = \operatorname{Mor}_{\mathcal{DGM}(A)}(M, N) \,/$ null homotopy.

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For every $\beta : M \longrightarrow N$, morphism of DG modules over a DG algebra A, we may consider a complex

$$\operatorname{Cone}\beta = \left(\Sigma M^{\natural} \oplus N^{\natural}, \begin{bmatrix} \partial^{\Sigma M} & 0 \\ \Sigma(\beta) & \partial^{N} \end{bmatrix} \right)$$

and a short exact sequence

$$0 \longrightarrow N \xrightarrow{\ \iota\ } \operatorname{Cone}\beta \xrightarrow{\ \pi\ } \Sigma M \longrightarrow 0.$$

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The category $\mathcal{H}(A)$ is triangulated, with shift functor $\overline{\Sigma}$ and with distinguished triangles the triangles in $\mathcal{H}(A)$ isomorphic (in $\mathcal{H}(A)$!) to

$$M \xrightarrow{\overline{f}} N \xrightarrow{\overline{\iota}} \operatorname{Cone}\beta \xrightarrow{\overline{\pi}} \overline{\Sigma} M \quad ,$$

for β morphism in $\mathcal{DGM}(A)$.

6. The derived category $\mathcal{D}(A)$

Given a complex M, one defines $Z(M)$, $B(M)$ and $H(M)$ as usual.

For A DG algebra, the homology defines a functor

$$H : \mathcal{DGM}(A) \longrightarrow \mathcal{GM}(H(A)).$$

A morphism $\beta : M \longrightarrow N$ in $\mathcal{DGM}(A)$ such that $H(\beta)$ is an isomorphism is called a *quasiisomorphism*.

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Quasiisomorphisms are well defined in $\mathcal{H}(A)$, and the set of morphsims

$$\{\text{quasiisomorphisms}\} \subseteq \{\text{morphisms in } \mathcal{H}(A)\}$$

is a multiplicative system in $\mathcal{H}(A)$.

So we may localise $\mathcal{H}(A)$ with respect to $\{\text{quasiisomorphisms}\}$.

Then $\mathcal{D}(A)$ is this localisation.

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